

THE CLASSIFICATION OF THREE-DIMENSIONAL GRADIENT LIKE MORSE-SMALE DYNAMIC SYSTEMS

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In papers [1 — 4] the topological classification of Morse-Smale vector fields on 2-manifolds and in [4,5] on 3-manifolds is got. In [6] classification three-dimensional Morse-Smale diffeomorphisms with non-intersected stable and unstable manifolds of saddle points is given.

In this paper new approach to the classifications problem of three-dimensional gradient like Morse-Smale dynamic systems is represented. The criterion of the such systems topological equivalence is in terms of homeomorphism between surfaces with two circles series on its is given.

The invariant of vector fields and diffeomorphisms are constructed and the classification of gradient like dynamic systems is obtained.

1. Basic definitions.

Smooth dynamic system (vector field or diffeomorphism) is called Morse-Smale system if:

- 1) it has the finite number of the critical elements (periodical trajectories for diffeomorphism, fixed points and the closed orbits in case of vector field) and all of them is non-degenerate (hyperbolic);
- 2) stable and unstable integral manifolds of critical elements have transversal intersections;
- 3) the limit set for every trajectory is critical element.

Heteroclinic trajectories of diffeomorphism are trajectories lying in the stable and unstable manifolds intersection for the same index critical elements.

Morse-Smale dynamic system is called by gradient like, if there isn't closed orbit in case of vector field [7] and heteroclinic trajectories in the case of diffeomorphism [8].

Vector fields are called topological equivalent, if there exists the homeomorphism of manifold onto itself, which maps integral trajectories into integral trajectories preserving their orientation. By graph we will understand finite 1-dimensional CW-complex. The isomorphism of graphs is the cell homeomorphism (id est., graph homeomorphism, which maps vertexes into vertexes and edges into edges).

2. The criterion of the vector fields topological equivalence.

Let M^3 be a closed oriented manifold, X and X' be Morse-Smale vector fields on it. Let a_1, \dots, a_k be fixed 0-points of the fields X , and a'_1, \dots, a'_k of the fields X' ; b_1, \dots, b_n and b'_1, \dots, b'_n be the fixed 1-points. Let K is the union of the 0- and 1-points stable manifolds. We shall consider tubular neighborhood $U(K)$ of this union.

We denote by $N = \partial U(K)$ the boundary of this neighborhood for field \tilde{O} and by N' for field X' . Then these boundaries are the surfaces which Heegaard splitting of manifold M^3 [9].

We denote by $v(x)$ and $u(x)$ stable and unstable manifolds of fixed points x . Let u_{-i} be the circles, which are obtained as a result of 1-points unstable manifold intersections. Then u_{-i} is a set of non crossed circles on surface F .

If c_1, \dots, c_m are the fixed 2-points, then intersection $v_i = v(c_i) \cap N$ will form another set of circles on surface F . Analogously, for the field X' on the surface N' there exists two sets of circles.

If there is one 0-point and one 3-point then sets of circles will be the systems of the meridians of surface which form Heegaard diagram of manifold M^3 [9].

Lemma 1. Field X is topological equivalent to X' if and only if there is a homeomorphism of surface $f : F \rightarrow F$, which maps the first set of circles into the first one and the second into the second.

Proof. Necessity results from building. We shall prove adequacy. Let such homeomorphism exists. We shall consider disks which lie onto unstable integral manifolds $U(b_i)$, contain dots b_i and bound by circles u_i . Then we can continue homeomorphisms from the boundary of these disks up to disks homeomorphisms and such that translate integral trajectories into integral trajectories (because each integral trajectory, except fixed dots, crosses the boundary of disk).

Analogously there exists disks homeomorphisms consisting of the of integral trajectories parts begun on the second type circles and ended in the fixed 2-point. Then surface F along with these disks cuts 3-manifold

on 3-disks, each of which has one fixed point of index 0 or 3. Having the homeomorphisms of the boundaries of these, shall continue their into interior of its. Thus we constructed homeomorphism of manifolds which set the topological equivalence of vector fields.

3. The expanding of the isomorphisms of graphs up to surfaces homeomorphism.

Let G is an oriented graph imbedded into surface F , and G' into F' . If the graphs are isomorphic (it is possible not preserving the orientation of edges) then there exist the finite number of the different isomorphisms between them. Then question on the existence of surfaces homeomorphism, restriction of which on graphs is the graphs isomorphism, is equivalent to question on the capability of the graphs isomorphism extension up to surfaces homeomorphism.

Let $g : G \rightarrow G'$ is a graphs isomorphism which maps vertex A_i of the graph G on vertex A'_i , and edges B_j on B'_j .

Let $g : G \rightarrow G'$ be a graph isomorphism which maps vertex A_i of graph G on vertex A'_i , and edges B_j on B'_j . Denote by $U(G)$ the tubular neighborhood of graph G in surface N and let be the projection of its closure on a graph. Then complement $N \setminus U(G)$ consist of surfaces F_i with boundary $\partial F_i = \partial U(G)$. Let us cut each circle from the boundary of surface F_i to arcs in a such way that each arc maps by projection p on one edge of graph G and reverse image $p^{-1}(B_j)$ for each edges consist of two arc from all surfaces. Let us choose such orientation of the arcs that the projection p preserves the orientation and we will denote these arcs by the same letters as the appropriate edges.

We fix the orientation on each surface F_i (which is compatible with surface F orientation if the surface F is oriented and in an arbitrary way otherwise). For each circle from the boundary of surface we form a word consisted from letters $B_j^{\pm 1}$ which denote arcs (edges of graph) from this circle. We write the letters in such consequence in which we meet it when we go around the circle along orientation compatible with surface F_i orientation. Letter has degree +1 if the orientation of correspondence arc is the same as the circle orientation and -1 otherwise. Two words are called equivalent if one from another can be obtained by cyclic letters permutation. This situation is obtained if we choose the other beginning of the circuit. The words are called reverse if one from another can be obtained in result of writing letters in reverse order with changing it degree and, possibly, cyclic permutation. This situation holds under the circle circuit with other orientation.

For each surface F_i we compose the list consisted of the number n_i which is equal the surface F_i genus and words which have written when we go around surface boundary along orientation. Two such list are called equivalent if it has the same numbers n_i and there is the one to one correspondence between words such that correspondent words are equivalent. The list are reverse if it has the same numbers n_i and all the words are reverse.

Thus for surface N and of graph G we construct the collection of lists in a way that each list correspond one surface F_i . Two such collection are called equivalent if there is one to one correspondence between list such that correspondent list are equivalent or reverse.

Lemma 2. Let G be the oriented graph embedded in surface N and G' in N' , $g : G \rightarrow G'$ be a graph isomorphism, which maps vertex A_i of graph G in vertex A'_i and edges B_j in B'_j . Then the graph isomorphism can be extended to surface homeomorphism iff replacing B_j on $B_j^{\pm 1}$ (we choose sign in depending on edges orientation preserving) in the lists collection for pair (N, G) we obtain the lists collection for pair (N', G') . In addition the homeomorphism preserve the orientation if all correspondent lists are equivalent.

Proof. Necessity of theorem condition is followed from construction. Indeed surfaces homeomorphism gives one to one correspondence between the lists and the word sets in its, in addition if we get start go around from other point we obtain equivalent words and lists, and if we reverse the orientation we obtain reverse ones.

Sufficiency. Let U_i be a connected component after cutting surface N by graph. Then they are homeomorphic to interiors of surfaces F_i . Surface N can be obtained in result of surfaces F_i gluing to graph G . Let us consider the boundaries of surfaces F_i as n -tagon (where n is the number of letters in the word). Each edge corresponds to one letter from word and attached map on one edge of graph G . Then the graph isomorphism and the word equivalence give natural homeomorphism between surface boundaries. Because of the genus and the number of boundary connected component are same for surfaces F_i and F'_i then they are

homeomorphic. In addition there exist homeomorphism, which expand the given boundary homeomorphism. This means that graph isomorphism can be expanded to surface homeomorphism.

4. The gradient like vector field invariant construction.

As we do it in section 2 for each vector field we construct surface with two sets of circles on it. The graph is the union of these circles. Vertexes of graph are the circle intersection and one arbitrary point on each circle without intersections. The edges are the arc between vertexes. All edges of graph are decomposed on two sets depending on which sets of circle the correspondent circle belong. We fix the arbitrary orientation of this graph. For this embedded graph as in section 3 we construct the word list collection with letters corresponded to the edges of the graph.

Definition. Such constructed graph with edges decomposition on two sets and with word list collection is called distinguished graph of vector field. Two distinguished graphs are called equivalent if there exist graph isomorphism, which preserve edges decomposition on two set. Replacing letters from word list collection of first graph by corresponding letters from second graph (with degree ± 1 depending on orientation) we have to obtain word list collection, which is equivalent to second graph word list collection.

Theorem 1. Two vector fields are topological equivalent if and only if their distinguished graphs are equivalent.

Proof is followed from using lemmas 1 and 2.

5. Topological conjugation of diffeomorphisms.

Let $f : M^3 \rightarrow M^3$ be a gradientlike Morse-Smale diffeomorphism. As it have been done in section 2 we construct surface with two set of circle on it and afterwards as in section 4 distinguished graph. Then the diffeomorphism f action on saddle point integral manifolds induce map between first type circles and map between second type circles and isomorphism of distinguished graph on itself, which we call by inner.

Theorem 2. Two gradientlike Morse-Smale diffeomorphisms f and g are topological conjugated iff there exist their distinguished graphs isomorphism which gives the equivalence between them and commutate with inner graph isomorphisms.

Proof. Necessity follow from construction. Let us prove the sufficiency. As in theorem 1 we construct homeomorphism h between manifolds which maps stable manifold on stable ones and unstable on unstable. In addition $h(f(U)) = g(h(U))$, where U is the part of stable or unstable manifold on which it separated by another manifolds. Analogously dimension 2 in [Grines] this homeomorphism can be corrected to needed homeomorphism.

6. Realization of dynamic system with given invariant.

Let us research the problem when distinguishing graph represent a surface with two sets of circles on them and gradient like vector field. Let K is a complex received in result of gluing surfaces, appropriate to the lists, to the graph. As each list of words sets a surface with boundary, structure of a surface (to be local homeomorphic to the plane) can be broken only in gluing points, that is on edges and in vertexes of the graph.

1) The condition that, that a complex K is locally plane in internal points of edges equivalent to that there is two part of surfaces boundaries that is glued to the edge. It means, that each letter or it reverses meets equally two times in all lists.

2) We shall assume that the first condition is executed. For each vertex we shall consider the set of incident edges. Two edges we shall name adjacent if they lie in the boundary of one of gluing surfaces and have the common vertex in it. This condition equivalent to that there is a word in which the appropriate letters are adjacent or are first and last letters of a word. Two edges we shall name equivalent if there is a chain of the adjacent edges, which connect them. Then a condition that that a complex K is local plane in vertex is equivalent to that for each vertex all incident edges are equivalent.

3) Let conditions 1) and 2) are executed. Then distinguishing graph determinate the graph on a surface. This graph is two sets of circles if and only if each vertex is incident to four edges or is the beginning and end of one edge, if it together with this edge forms a loop. Further, in vertex with four incident edges the adjacent edges should belong to various sets of edges (which correspond to two sets of circles).

Theorem 3. Distinguishing graph is a graph of gradient-like vector field if and only if

- a) Each letter in the whole set of the lists of words meets equally two times,
- b) Each vertex is incident to four edges or is the beginning and end of one edge,
- c) Each of four edges, is adjacent for two others from other set of edges.

Proof is followed from discussion above.

Literature.

1. Aranson S.H., Grines V.Z. Topological classification of flows on closed 2-manifolds // Advant. math sc.(Russian)- 41, N.1, 1986.- P.149-169.
2. A.V.Bolsinov, A.A.Oshemkov, V.V.Sharko. On Classification of Flows on Manifolds. I // Methods of Funct. An. And Topology, v.2, N.2, 1996, P.131 – 146.
3. Peixoto M. On the classification of flows on two-manifolds. In: Dynamical Systems, edited by M.Peixoto.- Academic Press.- 1973.- P.389-419.
4. Fleitas G. Classification of Gradient like flows of dimensions two and three // Bol. Soc. Brasil. Mat.-9, 1975.- P.155-183.
5. Umanski Ya.L. The circuit of three-dimensional Morse-Smale dynamic system without the closed trajectories //Izvestiya of USSR Acad. of Sc., 230, N6,1976, P.1286 – 1289.
6. Grines V.Z. Kalay I.N. Classification of three-dimensional gradient like dynamic systems // Advant. Math. Sc.(Russian), v.49, N.2, 1994.- P.149-150.
7. Smale. S. On gradient dynamical system // Ann. Math. 74, 1961, P.199-206.
8. Anosov D.V. Smooth dynamic systems 1 // Results of a science and engineering. Modern math. Problems. Fund. Directions, V.1, 1985,(Russian)- P.151-242.
9. Matveev S.V., Fomenko A.T. Algorithmic and computer methods in three-dimensional topology. — M.: MSU, 1991.- 301p.
10. Prishlyak A. On graph, which is embedded in a surface // Advant. Math. Sc. (Russian), V.56, N.4, 1997, -P.211-212.

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